Multi-dimensional Feedback Particle Filter for Coupled Oscillators

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Abstract—This paper presents a methodology for state estimation of coupled oscillators from noisy observations. The methodology is comprised of two parts: modeling and estimation. The objective of the modeling is to express dynamics in terms of the so-called phase variables. For nonlinear estimation, a coupled-oscillator feedback particle filter is introduced.

The filter is based on the construction of a large population of oscillators with mean-field coupling. The empirical distribution of the population encodes the posterior distribution of the phase variables. The methodology is illustrated with two numerical examples.

I. INTRODUCTION

This paper is concerned with the problem of estimating phases of coupled oscillators from noisy observation data. The motivation comes from biology where rhythmicity underlies several important behaviors – e.g., periodic gaits exhibited by animals during locomotion (walking, hopping, running etc); cf., [5]. The importance of the phase estimation problem is discussed in [13] and in several references therein.

An important motivation for us comes from the problem of controlling rhythmic behaviors – e.g., by using feedback control to stabilize walking gaits [4]. For such applications, estimation is seen as an intermediate step towards control.

The methodology proposed here comprises of two parts: 1. Modeling: The objective of the modeling is to obtain a reduced order model of the dynamics. The reduced order model comprises of weakly coupled oscillators, expressed in their phase variables; cf., [7], [8], [11].

As an example, consider a dynamical system with two weakly coupled oscillators: The reduced order model has two phase variables, \( \theta_1(t) \in [0,2\pi) \) and \( \theta_2(t) \in [0,2\pi) \), representing the instantaneous phase of the two oscillators at time \( t \). The dynamics evolve according to,

\[
\begin{align*}
\frac{d\theta_1(t)}{dt} &= \omega_1 + \epsilon I^{12}(\theta_1(t), \theta_2(t)) \quad (1a) \\
\frac{d\theta_2(t)}{dt} &= \omega_2 + \epsilon I^{21}(\theta_1(t), \theta_2(t)) \quad (1b)
\end{align*}
\]

where \( \omega_1, \omega_2 \) are the frequency parameters and \( I^{12}(\cdot), I^{21}(\cdot) \) are the interaction functions.

2. Estimation: The objective of the estimation problem is to construct a filter for estimating the instantaneous phase \( (\theta_1(t), \theta_2(t)) \) in the example above given a time-history of noisy observations. A Bayesian procedure is described for synthesis of the coupled oscillator feedback particle filter.

The filter comprises of a large population of oscillators coupled through a mean-field coupling. The empirical distribution of the population encodes the posterior distribution of the phase variables. The distribution is continuously updated based on noisy observations. Our approach is influenced by the conceptual framework in Lee and Mumford’s work [12], who proposed a Bayesian inference framework for the visual cortex based on particle filtering.

For the model described in (1a, 1b) the coupled oscillator particle filter is comprised of \( N \) stochastic processes \( \{ (\theta_1(i,t), \theta_2(i,t)) : 1 \leq i \leq N \} \), where the value \( (\theta_1(i,t), \theta_2(i,t)) \in [0,2\pi) \times [0,2\pi) \) is the state of the \( i \)-th particle at time \( t \). The dynamics evolve according to,

\[
\begin{align*}
\frac{d\theta_1(i,t)}{dt} &= \omega_1 + \epsilon I^{12}(\theta_1(i,t), \theta_2(i,t)) + dU_1^i(t) \quad (2a) \\
\frac{d\theta_2(i,t)}{dt} &= \omega_2 + \epsilon I^{21}(\theta_1(i,t), \theta_2(i,t)) + dU_2^i(t) \quad (2b)
\end{align*}
\]

where \( (U_1^i(t), U_2^i(t)) \) is the control input. It is chosen such that the empirical distribution of the population (for large \( N \)) approximates the posterior distribution of the phase variables \( (\theta_1(t), \theta_2(t)) \). A constructive design procedure for control synthesis is described, based on the feedback particle filter methodology [18], [17].

The two-state model is used here for illustrative purposes; the methodology is applicable to models with arbitrarily many phase variables.

The contributions of this paper are summarized as follows: The paper is amongst the first to explicitly use phase models to construct nonlinear estimators. This is important because a single oscillator is a model of the repetitive firing of a single neuron; cf., [7]. A coupled oscillator particle filter can thus be realized as a neuronal network. The activity of the network is encoded in the phase variables – e.g., \( \{\theta_1(i,t)\}_{i=1}^N \) and \( \{\theta_2(i,t)\}_{i=1}^N \) encode the posterior distribution of the (hidden) phase variables \( \theta_1(t) \) and \( \theta_2(t) \), respectively.

Apart from this paper, the problem of estimating phase variables appears in our own prior work [15], which applies the feedback particle filter to estimating a single phase variable associated with a walking gait cycle. In [13], a computational method, Phaser, is described based on taking a Hilbert transform and applying a Fourier series based correction (see also [6]). Dynamical systems techniques for recovering phase via delay coordinate embedding appear in [14], In [10], an approach for invariant reconstruction of phase equations from measurement data is discussed.

The outline of the remainder of this paper is as follows. The problem statement appears in Sec. II. The methodology is comprised of two parts, modeling and estimation, which are discussed in Sec. III-Sec. IV, respectively. Two illustrative numerical examples are described in Sec. V.
II. PROBLEM STATEMENT

A. Preliminaries and Notation

The dynamic model describes \( r \) weakly coupled oscillator systems. The state of the \( k \text{-th} \) system is denoted as \( X^k(t) \in \mathbb{R}^d \). The uncoupled dynamic model of the \( k \text{-th} \) system is given by,

\[
dX^k(t) = \tilde{a}^k(X^k(t))\,dt,
\]

(3)

where \( \tilde{a}^k \) is a \( C^1 \) function. The system is assumed to have an isolated asymptotically stable periodic orbit (limit cycle) with period \( 2\pi/\omega_k \). Denote the limit cycle solution as \( X^k_{LC}(\theta_k(t)) \) where \( \theta_k(t) = \theta_k(0) + \omega_k \cdot t \mod 2\pi \). Denote the set of points on the limit cycle as \( \mathcal{P}^k \).

By construction, the map \( X^k_{LC} : [0, 2\pi) \to \mathcal{P}^k \subset \mathbb{R}^d \) is invertible and denote the inverse map as \( \tilde{\phi}^k \). The inverse map \( \tilde{\phi}^k : \mathcal{P}^k \to [0, 2\pi) \) is defined as

\[
\tilde{\phi}^k(x) = \theta \quad \text{iff} \quad X^k_{LC}(\theta) = x,
\]

for \( x \in \mathcal{P}^k, \theta \in [0, 2\pi) \).

The inverse map is extended to a small neighborhood \( \mathcal{N}^k \) of the limit cycle \( \mathcal{P}^k \) by using the notion of asymptotic phase: Let \( \Phi^k : \mathcal{N}^k \times \mathbb{R} \to \mathcal{P}^k \) be the flow map so \( \Phi^k(x_0, t) \) is the solution of (3) from initial condition \( x_0 \in \mathcal{N}^k \). The asymptotic phase of the point \( x_0 \in \mathcal{N}^k \) is defined as the phase of \( x_0 \in \mathcal{P}^k \) such that \( ||\Phi^k(x_0, t) - \Phi(x_0, t)|| \to 0 \) as \( t \to \infty \). Denote the extended map as \( \phi^k \) where \( \phi^k : \mathcal{N}^k \to [0, 2\pi) \) and \( \phi^k = \tilde{\phi}^k \) on the periodic orbit \( \mathcal{P}^k \).

The instantaneous phase is defined in terms of the extended map:

\[
\theta_k(t) = \phi^k(X^k(t)), \quad \text{for} \quad X^k(t) \in \mathcal{N}^k.
\]

(4)

The set of points with same phase are called isochrons.

The phase model reduction is standard; cf., [1] and references therein.

**Example 1:** Consider the Hopf oscillator model. The state \( X(t) = (x(t), y(t)) \in \mathbb{R}^2 \) and the dynamics are given by

\[
dX(t) = \omega JX(t)\,dt + \gamma(1 - r^2)X(t)\,dt,
\]

(5)

where \( J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \), \( \omega = 1 \), \( \gamma = 0.1 \) and \( r^2 = (x(t))^2 + (y(t))^2 \).

The system has a limit cycle, \( X_{LC}(\theta) = (\cos(\theta), \sin(\theta)) \), where \( \theta = \omega t \mod 2\pi \). The set \( \mathcal{P} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \). For points \( (x, y) \in \mathcal{N} \), a small neighborhood of \( \mathcal{P} \), the phase \( \phi(x, y) = \arctan(y/x) \), i.e., the isochrons are rays originating at origin. Using (4), the instantaneous phase is given by

\[
\theta(t) = \phi(X(t)) = \arctan\left(\frac{y(t)}{x(t)}\right), \quad \text{for} \quad (x(t), y(t)) \in \mathcal{N}.
\]

**B. Problem Statement**

For the coupled system, the state is denoted as \( X(t) := (X_1(t), X_2(t), \ldots, X_r(t)) \). The dynamical system is given by,

\[
dX^k(t) = \tilde{a}^k(X^k(t))\,dt + \varepsilon \sum_{l=1}^{r} \tilde{b}^k(X^l(t), X^k(t))\,dt,
\]

(6)

where \( \tilde{a}^k, \tilde{b}^k \) are \( C^1 \) functions. The coupling parameter \( \varepsilon \) is assumed to be small enough such that \( X^k(t) \in \mathcal{N}^k \), a small neighborhood of the limit cycle \( \mathcal{P}^k \) of the uncoupled system, for all \( k = 1, 2, \ldots, r \) and \( t \geq 0 \).

The observation model is given by,

\[
dZ(t) = dZ(X(t))\,dt + \sigma_W\,dW(t),
\]

where \( Z(t) \in \mathbb{R}^m \) is the observation process, \( \{W(t)\} \) is a Wiener process, \( \sigma_W \in \mathbb{R} \) is the standard deviation parameter, and \( d\hat{h}(\cdot) \) is a \( C^1 \) function; \( \hat{h} \) is a column vector whose \( j \)-th coordinate is denoted as \( \hat{h}_j \) (i.e., \( \hat{h} = (\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_m)^T \)). By scaling, we assume without loss of generality that the covariance matrix associated with \( \{W(t)\} \) is an identity matrix.

The objective of the filtering problem is to estimate the posterior distribution of \( \theta(t) := (\theta_1(t), \theta_2(t), \ldots, \theta_r(t)) \) given the filtration (time history of observations) \( Z_{\{t\}} := \sigma(Z_s : 0 \leq s \leq t) \). Note that the phase \( \theta_k(t) \) is well-defined by (4) for \( X^k(t) \in \mathcal{N}^k \). The posterior is denoted by \( p^* \), so that for any measurable set \( A \subset [0, 2\pi)^r \),

\[
\int_{\theta \in A} p^*(\theta, t)\,d\theta = P\{\theta(t) \in A \mid Z_{\{t\}}\}.
\]

The filter is infinite-dimensional since it defines the evolution, in the space of probability measures, of \( \{p^*(\cdot, t) : t \geq 0\} \).

**Example 2:** Consider the coupled Hopf oscillator model of the form (6) with \( r = 2 \). The state \( X(t) = (X_1(t), X_2(t)) \) is four dimensional, with \( X_k(t) = (x_k(t), y_k(t)) \) for \( k = 1, 2 \). The dynamics of the coupled model is given by,

\[
\begin{align*}
    dX_1(t) &= (\omega JX_1(t) + \gamma(1 - r_1^2)X_1(t) + \varepsilon(X_1(t) - X_2(t)))\,dt, \quad (7a) \\
    dX_2(t) &= (\omega JX_2(t) + \gamma(1 - r_2^2)X_2(t) + \varepsilon(X_2(t) - X_1(t)))\,dt, \\
    & \quad (7b)
\end{align*}
\]

where \( r_1^2 = (x_1(t))^2 + (y_1(t))^2 \) for \( k = 1, 2 \), and other parameters are as in Example 1.

The observation process is two-dimensional \( Z(t) = (Z_1(t), Z_2(t)) \), and the observation model is given by,

\[
\begin{align*}
    dZ_1(t) &= x_1(t)\,dt + \sigma_W\,dW_1(t), \quad (8a) \\
    dZ_2(t) &= x_2(t)\,dt + \sigma_W\,dW_2(t), \quad (8b)
\end{align*}
\]

where \( W_1, W_2 \) are mutually independent standard Wiener processes, and \( \sigma_W \) is the standard deviation parameter.

We revisit this example in Sec. V-B.
III. MODELING: PHASE MODELS

A. Signal model

For the uncoupled limit cycling system (3), the state of the kth oscillator is \( \theta_k(t) \in [0, 2\pi) \), and the dynamics evolve according to,

\[
d\theta_k(t) = \omega_k \mod 2\pi.
\]

The solution of (3) is expressed as \( X_k(t) = X^k_{LC}(\theta_k(t)) \).

The modeling technique that is used in this paper is based on interpreting the original weakly coupled system (6), as a perturbation of this ideal uncoupled system. It can be shown that the approximation \( X_k(t) = X^k_{LC}(\theta_k(t)) + O(\varepsilon) \) holds for (6). That is, to \( O(\varepsilon) \), the effect of weak coupling is manifested through changes in the phase variable only.

The state of the reduced order model is \( \theta(t) = (\theta_1(t), \theta_2(t), \ldots, \theta_r(t)) \in [0, 2\pi)^r \), and the dynamics (to \( O(\varepsilon) \)) evolve according to,

\[
d\theta(t) = \omega_k + \varepsilon \sum_{k=1}^{r} f^{k}(\theta_k(t), \theta_j(t)) \mod 2\pi, \tag{9}
\]

for \( k = 1, 2, \ldots, r \) and \( t \geq 0 \). The interaction terms \( f^{k}(\theta_k, \theta_j) \) are obtained from the corresponding terms \( b^{k}(X_k, X_i) \) in (6). An algorithm for the same, based on a standard singular perturbation approach (see [1]), appears in Appendix VII.

Example 3: For the single-Hopf model (5) in Example 1, the state is \( \theta(t) \in [0, 2\pi) \) and the dynamics are given by,

\[
d\theta(t) = 1 \, dt \mod 2\pi.
\]

In terms of \( \theta(t) \), the (limit cycle) solution of (5) is given by \( X(t) = X_{LC}(\theta(t)) = (\cos(\theta(t)), \sin(\theta(t))) \).

For the coupled model (7a, 7b) in Example 2, the state \( \theta(t) = (\theta_1(t), \theta_2(t)) \in [0, 2\pi)^2 \) and the dynamics are given by,

\[
\begin{align*}
d\theta_1(t) &= 1 + \varepsilon \sin(\theta_1(t) - \theta_2(t)) \mod 2\pi, \\
d\theta_2(t) &= 1 + \varepsilon \sin(\theta_2(t) - \theta_1(t)) \mod 2\pi.
\end{align*}
\]

This reduced order phase model is derived as part of the Example 5 in Appendix VII. In terms of \( \theta(t) \), the (limit cycle) solution of the coupled full-order model (7a, 7b) is given by,

\[
\begin{align*}
X_1(t) &= X^1_{LC}(\theta_1(t)) + O(\varepsilon) = (\cos(\theta_1(t)), \sin(\theta_1(t))) + O(\varepsilon), \\
X_2(t) &= X^2_{LC}(\theta_2(t)) + O(\varepsilon) = (\cos(\theta_2(t)), \sin(\theta_2(t))) + O(\varepsilon).
\end{align*}
\]

B. Observation model

On the limit cycle, \( X_k(t) = X^k_{LC}(\theta_k(t)) + O(\varepsilon) \). We thus define an observation function on the limit cycle as,

\[
h(\theta) := \hat{h}(X_{LC}(\theta)),
\]

for \( \theta = (\theta_1, \theta_2, \ldots, \theta_r) \in [0, 2\pi)^r \) where \( X_{LC}(\theta) := (X^1_{LC}(\theta_1), X^2_{LC}(\theta_2), \ldots, X^r_{LC}(\theta_r)) \).

Example 4: Consider the observation model (8a, 8b) introduced in Example 2: the observation function

\[
\hat{h}((x_1, y_1), (x_2, y_2)) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]

So,

\[
h(\theta_1, \theta_2) = \hat{h}(X^1_{LC}(\theta_1), X^2_{LC}(\theta_2)) = \begin{bmatrix} \cos(\theta_1) \\ \cos(\theta_2) \end{bmatrix}.
\]

IV. ESTIMATION: FEEDBACK PARTICLE FILTER

Consider the multivariable filtering problem:

\[
\begin{align*}
d\theta(t) &= a(\theta(t)) \, dt + dB(t), \mod 2\pi, \tag{10a} \\
dZ(t) &= h(\theta(t)) \, dt + dW(t), \tag{10b}
\end{align*}
\]

where \( \theta(t) = (\theta_1(t), \theta_2(t), \ldots, \theta_r(t)) \in [0, 2\pi)^r \) is the state at time \( t \), \( Z(t) \in \mathbb{R}^{m} \) is the observation process, \( a(\cdot), h(\cdot) \) are \( 2\pi \)-periodic \( C^1 \) functions, and \( \{B(t)\}, \{W(t)\} \) are mutually independent Wiener processes of appropriate dimension. By scaling, we assume without loss of generality that the covariance matrices associated with \( \{B(t)\}, \{W(t)\} \) are identity matrices.

Recall here that our objective is to obtain the posterior distribution \( p_\theta(\theta \mid y) \) given the filtration \( \mathcal{F}_t \).

Remark 1: The signal model (10a) is allowed to be more general than (9) primarily to simplify the notation. Note, however, that the formulation allows one the option to include process noise in oscillators models.

A. Feedback Particle Filter

In our prior work [18], [17], [16], a feedback control-based approach to the solution of the nonlinear filtering problem was introduced. The resulting algorithm is referred to as the feedback particle filter.

The feedback particle filter is a controlled system. The state of the filter is \( \{\theta^i(t) : 1 \leq i \leq N\} \): The value \( \theta^i(t) = (\theta^i_1(t), \theta^i_2(t), \ldots, \theta^i_r(t)) \in [0, 2\pi)^r \) is the state for the \( i \)-th particle at time \( t \). Note that the \( i \)-th particle comprises of \( r \)-oscillators. The dynamics of the \( i \)-th particle have the following gain feedback form, expressed in its Stratonovich form,

\[
\begin{align*}
d\theta^i(t) &= a(\theta^i(t)) \, dt + dB^i(t) + K(\theta^i(t), t) \, dI^i(t), \mod 2\pi,
\end{align*}
\]

where \( \{B^i(t)\} \) are mutually independent standard Wiener processes, and \( I^i \) is similar to the innovation process that appears in the nonlinear filter,

\[
dI^i(t) := dZ(t) - \frac{1}{2} h(\theta(t)) \, dt,
\]

where \( \hat{h} := E[h(\theta(t)) \mid \mathcal{F}_t] \). In a numerical implementation, we approximate \( \hat{h} \approx N^{-1} \sum_{i=1}^{N} h(\theta^i(t)) := \hat{h}(N) \).

The gain function \( K \) is obtained as a solution to an Euler-Lagrange boundary value problem (E-L BVP): For \( j = 1, 2, \ldots, m \), the function \( \phi_j \) is a solution to the second-order differential equation,

\[
\begin{align*}
\nabla \cdot (p(\theta, t) \nabla \phi_j(\theta, t)) &= -(h_j(\theta) - \hat{h}_j)p(\theta, t), \\
\int_{[0, 2\pi)^r} \phi_j(\theta, t)p(\theta, t) \, d\theta &= 0,
\end{align*}
\]

2423
where $p$ denotes the conditional distribution of $\theta^j(t)$ given $Z_t$. In terms of these solutions, the gain function is given by,
\[ [K]_{ij}(\theta, t) = \frac{\partial \phi_j}{\partial \theta_i}(\theta, t). \]

Note that the gain function needs to be obtained for each value of time $t$.

It is shown in [17] that the feedback particle filter is consistent with the nonlinear filter, given consistent initializations $p^*(\cdot, 0) = p^*(\cdot, 0)$. Consequently, if the initial conditions $\{\theta(0)\}_{i=1}^N$ are drawn from the initial distribution $p^*(\cdot, 0)$ of $\theta(0)$, then, as $N \to \infty$, the empirical distribution of the particle system approximates the posterior distribution $p^*(\cdot, t)$ for each $t$ (see [16], [17] for details).

**Notation:** $L^2([0, 2\pi)^r; p)$ is used to denote the Hilbert space of $2\pi$-periodic functions on $[0, 2\pi)^r$ that are square-integrable with respect to density $p^*(\cdot, t)$; $H^1([0, 2\pi)^r; p)$ is used to denote the Hilbert space of functions whose first $k$-derivatives (defined in the weak sense) are in $L^2([0, 2\pi)^r; p)$.

Denote
\[ H_0^1([0, 2\pi)^r; p) := \left\{ \phi \in H^1 \left| \int \phi(\theta) p(\theta, t) d\theta = 0 \right. \right\}. \]

A function $\phi_j \in H_0^1([0, 2\pi)^r; p)$ is said to be a weak solution of the BVP (11) if
\[ \int \nabla \phi_j(\theta, t) \cdot \nabla \psi(\theta) p(\theta, t) d\theta = \int (h_j(\theta) - \hat{h}_j) \psi(\theta) p(\theta, t) d\theta, \]
for all $\psi \in H^1([0, 2\pi)^r; p)$.

Denoting $E[|\cdot| := f \cdot p(\theta, t) d\theta$, the weak form of the BVP (11) can also be expressed as follows:
\[ E[\nabla \phi_j, \nabla \psi] = E[(h_j - \hat{h}_j) \psi], \quad \forall \psi \in H^1([0, 2\pi)^r; p). \] (12)

This representation is useful for the numerical algorithm described next.

**B. Synthesis of Gain Function**

In this section, a Galerkin algorithm is described to obtain an approximate solution of (11). Since there are $m$ uncoupled BVPs, without loss of generality, we assume scalar-valued observation in this section, with $m = 1$, so that $K = \nabla \phi$.

The time $t$ is fixed. The explicit dependence on time is suppressed for notational ease (That is, $p(\theta, t)$ is denoted as $p(\theta)$, $\phi(\theta, t)$ as $\phi(\theta)$ etc.).

The function $\phi$ is approximated as,
\[ \phi(\theta) = \sum_{i=1}^L \kappa_i \psi_i(\theta), \]

where $\{\psi_i(\theta)\}_{i=1}^L$ are the Fourier basis functions on $[0, 2\pi)^r$. The gain function $K$ is given by,
\[ K(\theta) = \nabla \phi(\theta) = \sum_{i=1}^L \kappa_i \nabla \psi_i(\theta). \]

The finite-dimensional approximation of (12) is to choose constants $\{\kappa_i\}_{i=1}^L$ such that
\[ \sum_{i=1}^L \kappa_i E[\nabla \psi_i \cdot \nabla \psi] = E[(\hat{h} - \hat{h}) \psi], \quad \forall \psi \in S, \] (13)

where $S := \text{span}\{\psi_1, \psi_2, \ldots, \psi_L\} \subset H^1([0, 2\pi)^r; p)$.

Denoting $[A]_{ij} = E[\nabla \psi_i \cdot \nabla \psi_j]$, $b_k = E[(\hat{h} - \hat{h}) \psi_k]$, $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_L)^T$, the finite-dimensional approximation (13) is expressed as a linear matrix equation:
\[ A \kappa = b. \] (14)

The matrix $A$ and vector $b$ are approximated by using only the particles:
\[ [A]_{ij} = E[\nabla \psi_i \cdot \nabla \psi_j] \approx \frac{1}{N} \sum_{i=1}^N \nabla \psi_i(\theta^i(t)) \cdot \nabla \psi_j(\theta^i(t)), \]
\[ b_k = E[(\hat{h} - \hat{h}) \psi_k] \approx \frac{1}{N} \sum_{i=1}^N (h(\theta^i(t)) - \hat{h}) \psi_k(\theta^i(t)), \]

where $\hat{h} \approx N^{-1} \sum_{i=1}^N h(\theta^i(t))$.

The important point to note is that the gain function is expressed in terms of averages taken over the population.

**V. Examples**

**A. Single Hopf oscillator**

Consider the filtering problem for the single-Hopf model (see Example 1):
\[ dX(t) = \omega JX(t) dt + \gamma(1 - r^2) X(t), \]
\[ dZ(t) = \tilde{h}(X(t)) dt + \sigma_W dW(t), \]

where $X(t) = (x(t), y(t)) \in \mathbb{R}^2$, $\gamma = x^2(t) + y^2(t)$, and parameters $\omega = 1$, $\gamma = 0.1$ and $\sigma_W = 0.1$. Fig. 1 depicts the trajectory $x(t)$ and noisy observations $Y(t)$. At each discrete time, these are obtained as $Y(t) := \Delta z_{t}/\Delta t$, where $\Delta t$ denotes the discrete time step, and $\Delta z_{t} = Z_t + \Delta t - Z(t)$. In numerical simulations, $\Delta t = 0.01$ is chosen.

Fig. 1. Observations over one cycle for example 1.

Recall that the limit cycle solution is given by $X(t) = X_{LC}(\theta(t)) = (\cos(\theta(t)), \sin(\theta(t)))$, where $\theta(t) = \theta(0) + \omega t$ mod $2\pi$. The objective is to approximate the posterior of the instantaneous phase $\theta(t)$ given $Z_t$. 

\[ \text{Fig. 1. Observations over one cycle for example 1.} \]
The feedback particle filter is given by,
\[
d\theta^i(t) = \omega_i \, dt + \sigma_B \, dB^i(t) + K(\theta^i(t), t) \circ \left( dZ(t) - \frac{h(\theta^i(t)) + \hat{h}}{2} \right), \quad \text{mod } 2\pi,
\]
where \( h(\theta^i(t)) = \cos(\theta^i(t)) \) and \( \hat{h} = N^{-1} \sum_{j=1}^N \cos(\theta^j(t)) \).

For the exact filter, \( \omega = \omega_0 \) and \( \sigma_B = 0 \). In numerical simulations, we include a small noise \( \sigma_B > 0 \) and pick \( \omega_t \) from a uniform distribution on \( [\omega - \delta, \omega + \delta] \). Such a model is more realistic from an implementation standpoint, even though it is not exact.

The gain function \( K(\theta, t) = \frac{\partial \phi}{\partial \theta}(\theta, t) \) is obtained via the solution of the E-L equation:
\[
\frac{\partial}{\partial \theta} \left( p(\theta, t) \frac{\partial \phi}{\partial \theta}(\theta, t) \right) = -\left( h(\theta) - \hat{h} \right) p(\theta, t).
\]

An approximate solution is obtained by using the Galerkin method described in Sec. IV-B with basis functions \( \{\cos(\theta), \sin(\theta)\} \). In terms of basis functions,
\[
K(\theta, t) = -\kappa_1(t) \sin(\theta) + \kappa_2(t) \cos(\theta),
\]
where \( \kappa(t) := [\kappa_1(t), \kappa_2(t)] \) are obtained at each time by solving the linear matrix equation (see (14)):
\[
\begin{bmatrix}
\frac{1}{2} \sum \sin^2(\theta^i(t)) & -\frac{1}{2} \sum \sin(\theta^i(t)) \cos(\theta^i(t)) \\
-\frac{1}{2} \sum \sin(\theta^i(t)) \cos(\theta^i(t)) & \frac{1}{2} \sum \cos^2(\theta^i(t))
\end{bmatrix}
\begin{bmatrix}
\kappa_1(t) \\
\kappa_2(t)
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{2} \sum (h(\theta^i(t)) - \hat{h}) \sin((\theta^i(t))) \\
\frac{1}{2} \sum (h(\theta^i(t)) - \hat{h}) \cos((\theta^i(t)))
\end{bmatrix}.
\]

We next discuss the result of numerical experiments. The particle filter model is given by (16) with gain function \( K(\theta^i(t), t) \), obtained using solution of (17). The number of particles \( N = 1,000 \) and their initial condition \( \{\theta^i(0)\}_{i=1}^N \) was sampled from a uniform distribution on circle \([0, 2\pi] \). The noise parameter \( \sigma_B = 0.1 \) and the frequencies \( \omega_t \) are sampled from a uniform distribution on \([1 - \delta, 1 + \delta] \) with \( \delta = 0.5 \).

Fig. 2 summarizes some of the results of the numerical simulation. The initialization due to the initial condition converges rapidly as shown in part (a). A single cycle from a time-window after transients have converged is depicted in part (b); the figure compares the sample path of the actual state \( \theta(t) \) (as a solid line) with the estimated mean \( \hat{\theta}^{(N)}(t) \) (as a dashed line). The shaded area indicates \( \pm 2 \) standard deviation bounds. The density at two time instants is depicted in part (c). These results show that the filter is able to track the phase accurately.

Note that at each time instant \( t \), the estimated mean, the bounds and the density \( p(\theta, t) \) shown here are all approximated from the ensemble \( \{\theta^i(t)\}_{i=1}^N \).

B. Coupled Hopf oscillator

Consider next the filtering problem for the coupled Hopf model. The model is described in Example 2: The signal model is the 4-dimensional coupled Hopf model (7a, 7b) and the observation model (8a, 8b) is two-dimensional.

Starting from an arbitrary initial condition, one asymptotically obtains an anti-phase limit cycle solution where \( X_k(t) := X_{LC}(\theta_k(t)) = (\cos(\theta_k(t)), \sin(\theta_k(t))) \) for \( k = 1, 2 \) and \( \theta_1(t) \) and \( \theta_2(t) \) are out of phase. Fig. 3 (a) depicts this solution for \( \epsilon = 0.05 \).

The objective is to approximate the posterior of the phase \( \theta_1(t), \theta_2(t) \) given \( \theta_t^i \).

The feedback particle filter is given by,
\[
d\theta_1^i(t) = \omega_1 \, dt + \epsilon \sin(\theta_1^i(t) - \theta_2^i(t)) \, dt + \sigma_B \, dB_1^i(t)
\]
\[
+ K_1(\theta_1^i(t), t) \circ \left( dZ(t) - \frac{h_1(\theta_1^i(t)) + \hat{h}_1(t)}{2} \right), \quad \text{mod } 2\pi,
\]
\[
d\theta_2^i(t) = \omega_2 \, dt + \epsilon \sin(\theta_1^i(t) - \theta_2^i(t)) \, dt + \sigma_B \, dB_2^i(t)
\]
\[
+ K_2(\theta_1^i(t), t) \circ \left( dZ(t) - \frac{h_2(\theta_1^i(t)) + \hat{h}_2(t)}{2} \right), \quad \text{mod } 2\pi,
\]
where \( \theta^i(t) = (\theta_1^i(t), \theta_2^i(t)) \in [0, 2\pi]^2 \),
\[
h(\theta^i(t)) = \begin{bmatrix}
h_1(\theta_1^i(t)) \\
h_2(\theta_2^i(t))
\end{bmatrix} = \begin{bmatrix}
\cos(\theta_1^i(t)) \\
\cos(\theta_2^i(t))
\end{bmatrix},
\]
and \( \hat{h}(t) = N^{-1} \sum_{i=1}^N \cos(\theta^i(t)) \). As in the preceding section, we include small noise \( \sigma_B > 0 \) and pick frequencies \( \omega_1, \omega_2 \) from a uniform distribution on \([\omega - \delta, \omega + \delta] \).

The gain function,
\[
\begin{bmatrix}
K_1(\theta_1, \theta_2, t) \\
K_2(\theta_1, \theta_2, t)
\end{bmatrix} = \begin{bmatrix}
\frac{\partial h_1}{\partial \theta_1}(\theta_1, \theta_2, t) \\
\frac{\partial h_1}{\partial \theta_2}(\theta_1, \theta_2, t)
\end{bmatrix}, \quad \begin{bmatrix}
\frac{\partial h_2}{\partial \theta_1}(\theta_1, \theta_2, t) \\
\frac{\partial h_2}{\partial \theta_2}(\theta_1, \theta_2, t)
\end{bmatrix}.
\]
where each column is obtained by solving the E-L equation:
\[ \nabla \cdot (p(\theta,t) \nabla \phi_j(\theta,t)) = -(h_j(\theta) - \hat{h}_j)p(\theta,t), \quad \text{for } j = 1, 2. \]

An approximate solution is obtained by using the Galerkin method described in Sec. IV-B with basis functions \( \{\sin(\theta_1), \cos(\theta_1), \sin(\theta_2), \cos(\theta_2)\} \). This leads to a \( 4 \times 4 \) linear matrix problem (14), whose solution is obtained numerically. As in the preceding section, the calculation of the gain function requires averages taken over the population.

We next discuss the result of numerical experiments. The number of particles \( N = 1,000 \) and their initial condition \( \{\theta(0)\}_{i=1}^{N} \) and \( \{\tilde{\theta}(0)\}_{i=1}^{N} \) was sampled from a uniform distribution on the torus \([0, 2\pi]^2\). The noise parameter \( \sigma_R = 0.1 \) and frequencies are sampled uniformly from \([0.5, 1.5]\).

Fig. 3 (b) compares the sample path of the actual state \( (\theta_1(t), \theta_2(t)) \) (as a solid line) with the estimated mean \( \langle \hat{\theta}^{(N)}_1(t), \hat{\theta}^{(N)}_2(t) \rangle \) (as a dashed line). The standard deviation bounds are tight: the density at a single time instant is depicted in part (c). These results show that the filter is able to track the phase variables on the torus accurately.

VI. CONCLUSIONS

The application of singular perturbation techniques to obtain a reduced order model provides an approach to nonlinear filter design for a coupled oscillator model. This technique is general, and is expected to find many applications.

The importance of coupled oscillators as models of both movement in animals (periodic gaits), and of central pattern generators in nervous system provided the impetus to this work; cf. [11], [3], [9], [2]. The filtering viewpoint espoused in this paper is useful because it connects naturally to feedback control of periodic behavior. This is a subject of ongoing work.

In addition, we are currently applying these techniques to phase estimation in the power grid, which is of significant interest in industry. Reliable power system operations require the balance of supply and demand on a second-by-second basis: The frequency must be kept at 60Hz, and consistency of the phase of generation at neighboring nodes in the grid must be maintained. Control requires state estimation of precisely the kind described in this paper.

VII. APPENDIX: MODEL REDUCTION

In this section, an algorithm is described to obtain the reduced order phase model (9) starting from the weakly coupled dynamical system model (6). We use the notation of Sec. II-A.

For the \( k \)-th system (3), the phase response curve (PRC), denoted as \( P_k(\theta) \), is defined to be the \( 2\pi \)-periodic solution of the adjoint equation:
\[ \omega_k \frac{dP_k}{d\theta}(\theta) + (Da^k(X_{LC}^k(\theta)))^T P_k(\theta) = 0, \quad \text{for } \theta \in [0, 2\pi), \]
where \( Da^k(X_{LC}^k(\theta)) \) is the Jacobian of the vector-field \( a^k \) evaluated on the limit cycle \( X_{LC}^k(\theta) \), and \( P_k(\theta) \) satisfies the normalizing condition \( \langle P_k(0), a^k(X_{LC}^k(0)) \rangle = 1 \), where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^d \).

In terms of PRC, the interaction term in (9) is obtained as
\[ \ell_{k\ell}(\theta_k, \theta_\ell) = \langle P_k(\theta_k), b^\ell(X_{LC}^k(\theta_k), X_{LC}^\ell(\theta_\ell)) \rangle. \quad (19) \]

The derivation relies on a singular perturbation argument; cf., [7], [8], [1].

**Example 5:** Consider the single-Hopf model (5) in Example 1. The limit cycle solution is given by \( X_{LC}^1(\theta) = (\cos(\theta), \sin(\theta)) \), and \( a^1(X_{LC}^1(\theta)) = (-\sin(\theta), \cos(\theta)) \). The adjoint equation is
\[ \frac{dP_1}{d\theta}(\theta) + AP_1(\theta) = 0, \]
where
\[ A = \begin{bmatrix} -2\cos^2(\theta) & 1 - 2\sin(\theta)\cos(\theta) \\ -1 - 2\sin(\theta)\cos(\theta) & -2\sin^2(\theta) \end{bmatrix}, \]
and the normalizing constraint
\[ \langle P_1(0), a^1(X_{LC}^1(0)) \rangle = \langle P_1(0), (0,1) \rangle = 1. \]

The solution to the adjoint equation, the phase response curve
\[ P_1(\theta) = (-\sin(\theta), \cos(\theta)). \]

Consider now the coupled model (7a, 7b) in Example 2 with \( b^{12} = X_1 - X_2 \). Using (19), the interaction term for the
phase model is obtained as,

\[ I_{12}(\theta_1, \theta_2) = \begin{bmatrix} -\sin(\theta_1) & \cos(\theta_1) \\ \cos(\theta_1) & -\cos(\theta_2) \\ \sin(\theta_1) & -\sin(\theta_2) \end{bmatrix} = \sin(\theta_1 - \theta_2). \]

References


